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Research on a Special Fractional Integral

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Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we use some methods to find a special fractional integral. In fact, our result is a generalization of ordinary calculus result.**

Keyword: **New multiplication, fractional analytic functions, special fractional integral.**

I. INTRODUCTION

Fractional calculus is the theory of derivative and integral of non-integer order, which can be traced back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. Fractional calculus has been attracting the attention of scientists and engineers from long time ago, and has been widely used in physics, engineering, biology, economics and other fields [1-15]. The definition of fractional derivative is not unique. The commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [16-19]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this paper, based on a new multiplication of fractional analytic functions, we use some techniques to find the following special integral:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\right]^{\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)}\right],$$

where $0<\alpha\leq 1$. Moreover, our result is a generalization of classical calculus result.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([20])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{2}$$

where $\Gamma(\ )$ is the gamma function.

Next, we introduce the definition of fractional analytic function.

Definition 2.2 ([21])**:** If x, x_0, and a_n are real numbers for all n, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.3 ([22])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}, \tag{3}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} . \tag{4}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) (x - x_0)^{n\alpha}. \end{aligned} \tag{5}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} . \end{aligned} \tag{6}$$

Definition 2.4 ([23])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} , \tag{7}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} . \tag{8}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{9}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!} \big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{10}$$

Definition 2.5 ([24])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} , \tag{11}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} . \tag{12}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{13}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!} \big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{14}$$

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Definition 2.6 ([25])**:** If $0 < \alpha \le 1$, then the α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}, \tag{15}$$

Notation 2.7: If r is a real number and n is a positive integer. Define $(r)_n = r(r-1)\cdots(r-n+1)$, and $(r)_0 = 1$.

III. MAIN RESULTS

In this section, we find a special fractional integral by using some methods. At first, a lemma is needed.

Lemma 3.1: *Assume that* $0 < \alpha \le 1$, *and* n *is a non-negative integer. Then*

$$\left({}_0I_x^\alpha\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha n}[Ln_\alpha(x^\alpha)]^{\otimes_\alpha n}\right] = \left[\sum_{m=0}^{n} \frac{(-1)^m (n)_m}{(n+1)^{m+1}}[Ln_\alpha(x^\alpha)]^{\otimes_\alpha (n-m)}\right]\otimes_\alpha \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha (n+1)}. \tag{16}$$

Proof Let $\frac{1}{\Gamma(\alpha+1)}x^\alpha = E_\alpha(t^\alpha)$, then $\frac{1}{\Gamma(\alpha+1)}t^\alpha = Ln_\alpha(x^\alpha)$, and hence

$$\left({}_0I_x^\alpha\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha n}[Ln_\alpha(x^\alpha)]^{\otimes_\alpha n}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha n}[Ln_\alpha(x^\alpha)]^{\otimes_\alpha n}\otimes_\alpha \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]$$

$$= \left({}_0I_t^\alpha\right)\left[[E_\alpha(t^\alpha)]^{\otimes_\alpha n}\left[\frac{1}{\Gamma(\alpha+1)}t^\alpha\right]^{\otimes_\alpha n}\otimes_\alpha E_\alpha(t^\alpha)\right]$$

$$= \left({}_0I_t^\alpha\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}t^\alpha\right]^{\otimes_\alpha n}\otimes_\alpha [E_\alpha(t^\alpha)]^{\otimes_\alpha (n+1)}\right]$$

$$= \left({}_0I_t^\alpha\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}t^\alpha\right]^{\otimes_\alpha n}\otimes_\alpha E_\alpha((n+1)t^\alpha)\right]$$

$$= \left[\sum_{m=0}^{n} \frac{(-1)^m (n)_m}{(n+1)^{m+1}}\left[\frac{1}{\Gamma(\alpha+1)}t^\alpha\right]^{\otimes_\alpha (n-m)}\right]\otimes_\alpha E_\alpha((n+1)t^\alpha)$$

$$= \left[\sum_{m=0}^{n} \frac{(-1)^m (n)_m}{(n+1)^{m+1}}\left[\frac{1}{\Gamma(\alpha+1)}t^\alpha\right]^{\otimes_\alpha (n-m)}\right]\otimes_\alpha [E_\alpha(t^\alpha)]^{\otimes_\alpha (n+1)}$$

$$= \left[\sum_{m=0}^{n} \frac{(-1)^m (n)_m}{(n+1)^{m+1}}[Ln_\alpha(x^\alpha)]^{\otimes_\alpha (n-m)}\right]\otimes_\alpha \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha (n+1)}.$$

q.e.d.

Theorem 3.2: *If* $0 < \alpha \le 1$, *then*

$$\left({}_0I_x^\alpha\right)\left[\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]^{\otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{m=0}^{n} \frac{(-1)^m (n)_m}{(n+1)^{m+1}}[Ln_\alpha(x^\alpha)]^{\otimes_\alpha (n-m)}\right]\otimes_\alpha \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha (n+1)}. \tag{17}$$

Proof

$$\left({}_0I_x^\alpha\right)\left[\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]^{\otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[E_\alpha\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes_\alpha Ln_\alpha(x^\alpha)\right)\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes_\alpha Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha n}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha n}[Ln_\alpha(x^\alpha)]^{\otimes_\alpha n}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left({}_0I_x^{\alpha} \right) \left[\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes_\alpha n} [Ln_\alpha(x^\alpha)]^{\otimes_\alpha n} \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{m=0}^{n} \frac{(-1)^m (n)_m}{(n+1)^{m+1}} [Ln_\alpha(x^\alpha)]^{\otimes_\alpha (n-m)} \right] \otimes_\alpha \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes_\alpha (n+1)} . \qquad \text{q.e.d.}$$

IV. CONCLUSION

In this paper, we use some methods to obtain a special fractional integral based on a new multiplication of fractional analytic functions. Moreover, our result is a generalization of classical calculus result. In the future, we will continue to use our methods to solve the problems in engineering mathematics and fractional differential equations.

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